Design Derivatives of Eigenvalues and Eigenfunctions for Self-Adjoint Distributed Parameter Systems

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In this paper, analytic expressions are obtained for the design derivatives of eigenvalues and eigenfunctions of self-adjoint linear distributed parameter systems. Explicit treatment of boundary conditions is avoided by casting the eigenvalue equation into integral form. Results are expressed in terms of the linear operators defining the eigenvalue problem, and are therefore quite general. Sufficiency conditions appropriate to structural optimization of eigenvalues are obtained.

Introduction

DERIVATIVES of eigenvalues and eigenvectors with respect to the design parameters are useful, if not essential, for design sensitivity and structural optimization studies. In one of the earlier works to address this question, Fox and Kapoor[1] restrict their discussion to self-adjoint discrete systems. Plaut and Huseyin[2] extend this approach to non-self-adjoint discrete systems by treating the associated adjoint eigenvalue problem.

For distributed parameter systems, design derivatives of eigenvalues were first encountered in optimization studies. Prager and Taylor[3] generalized several earlier studies[4,5] using Rayleigh's principle. This approach, however, is not suitable for determining the design derivative of eigenfunctions, or for calculating higher order derivatives of eigenvalues. Finally, Rayleigh's principle is not appropriate for the determination of conditions, necessary or sufficient, to ensure the existence of the design derivatives of eigenvalues and eigenfunctions.

The foregoing considerations were recently addressed by Haug and Rousselet[6] using a functional analysis approach. In this paper, a relatively simple method is used to determine explicit results for the design derivatives of eigenvalues and eigenfunctions. Self-adjoint operator equations are cast into their integral form using Green's function. Since the design derivative of Green's functional is explicitly known,[7] it is a straightforward matter to differentiate the eigenvalue equation with respect to the design parameters to obtain an equation for the variations of the eigenvalues and eigenfunctions. The solution to this equation is explicitly obtained by recognizing that the eigenfunctions form a complete set.

Problem Definition

The closed, bounded and regular (in the sense of Kellogg) domain of an elastic structure is denoted by $\bar{\Omega}$ for which the interior is Ω and boundary is $\partial\Omega$. Furthermore, $\partial\Omega$ is the union of two complementary regular subsets $\partial\Omega_1$ and $\partial\Omega_2$. A typical point in $\bar{\Omega}$ is denoted by x, and the design variable(s) for the structure are collectively denoted by $S(x)$.

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The class of eigenvalue problems under discussion are defined through the operator equation

$$T^*E(S)Tu_i=\lambda_i M(S)u_i \quad x\epsilon\Omega \tag{1a}$$

$$Bu_i=0 \quad x\epsilon\partial\Omega_1 \tag{1b}$$

$$B^*E(S)Tu_i=0 \quad x\epsilon\partial\Omega_2 \tag{1c}$$

Here, T and T^* are L_2 adjoint differential operators, and B and B^* the corresponding adjoint boundary operators. It is important to note that all of the operators T, T^*, B, and B^* are independent of S. Consequently, the important class of problems that require shape variations are explicitly excluded from this study. Dependence of the eigenvalues λ_i and eigenfunctions u_i on S results solely from the design dependence of the self-adjoint stiffness operator $E(S)$ and self-adjoint mass operator $M(S)$.

It is observed that Eqs. (1) characterize the frequency response of linear elastic systems. The eigenvalue problem associated with elastic buckling can be obtained formally from Eqs. (1) by deleting the dependence of M on S. It is convenient to recast Eqs. (1) into an integral formulation using Green's function G corresponding to the operators on the left-hand side of Eqs. (1). Thus the eigenvalue problem becomes

$$u_i(y)=\lambda_i(G(\cdot,y),Mu_i)_\Omega \tag{2}$$

Here $(\cdot,\cdot)_\Omega$ denotes the L_2 inner product, and Ω its domain of integration.

It was observed previously that the eigenfunctions u_i and their corresponding eigenvalues λ_i are functionals of the design. Thus

$$u_i=u_i(S) \qquad \lambda_i=\lambda_i(S)$$

Consequently, if δS denotes a small change in the design variable S, then the eigenfunction u_i will change by an amount Δu_i, where

$$\Delta u_i=u_i(S+\delta S)-u_i(S) \tag{3}$$

If only terms linear in δS are retained in Eq. (3), Δu_i becomes the first variation (or, equivalently, the Gateaux differential) and is denoted by δu_i. The primary object of this study is to develop explicit expressions to evaluate the first variations δu_i and $\delta\lambda_i$ as well as the higher order variations.

The Design Derivatives

First Variation

According to Eq. (2), the first variation of the eigenfunction u_i satisfies the integral equation

$$\delta u_i = \lambda_i(\delta G, Mu_i)_\Omega + \lambda_i(G,\delta M u_i)_\Omega + \lambda_i(G,M\delta u_i)_\Omega + \delta\lambda_i(G,Mu_i)_\Omega \tag{4}$$

It has been previously shown[7] that the variation δG satisfies

$$\delta G(y,z) = -(TG(\cdot,y),\delta ETG(\cdot,z))_\Omega \tag{5}$$

After substituting Eq. (5) into the first term on the right-hand side of Eq. (4), changing the indicated order of the resulting integrations and then employing Eq. (2), the term $-(TG,\delta ETu_i)_\Omega$ results.

Equation (4) is therefore equivalent to

$$\delta u_i - \lambda_i(G,M\delta u_i)_\Omega = F_i \tag{6a}$$

where

$$F_i = \lambda_i^{-1}\delta\lambda_i u_i + \lambda_i(G,\delta Mu_i)_\Omega - (TG,\delta ETu_i)_\Omega \tag{6b}$$

Since F_i does not depend upon δu_i, Eq. (6a) will be recognized as an integral equation of the second kind for δu_i. According to Fredholm's Theorem, a solution δu_i of Eq. (6a) will exist if, and only if, F_i is orthogonal to every eigenfunction corresponding to the eigenvalue λ_i.

Distinct Eigenvalues

Since u_i is the only eigenfunction corresponding to λ_i, the necessary and sufficient condition for the existence of δu_i is

$$(F_i,Mu_i)_\Omega = 0 \tag{7}$$

By taking the inner product of the left-hand side of Eq. (6a) with Mu_i, changing the indicated order of integrations and then substituting Eq. (2) into the resulting expression, Eq. (7) will be verified. Then, substitution of Eq. (6b) into Eq. (7) will determine $\delta\lambda_i$. Thus

$$\lambda_i^{-1}\delta\lambda_i(u_i,Mu_i)_\Omega + \lambda_i((G,\delta Mu_i)_\Omega,Mu_i)_\Omega - ((TG,\delta ETu_i)_\Omega,Mu_i)_\Omega = 0 \tag{8}$$

Again, if the orders of integration are changed in the second and third terms of Eq. (8), and Eq. (2) substituted into the result, it is found that

$$\delta\lambda_i = \frac{(Tu_i,\delta ETu_i)_\Omega - \lambda_i(u_i,\delta Mu_i)_\Omega}{(u_i,Mu_i)_\Omega} \tag{9}$$

Equation (9) could have been obtained in a more straightforward manner using Rayleigh's principle. Indeed, Prager and Taylor[3] obtained Eq. (9) for the special case where δE and δM were both proportional to δS. Rayleigh's principle however, is not useful for determining δu_i or higher variations such as $\delta^2\lambda_i$.

For definitiveness, the eigenfunctions are normalized with respect to the mass operator. Therefore the orthonormality conditions become

$$(u_i,Mu_j)_\Omega = \delta_{ij} \tag{10}$$

It remains to determine δu_i. Following earlier approaches[2] in which the completeness of the eigenfunctions is recognized, the variations δu_i and F_i are expanded in terms of the eigenfunctions. Therefore

$$\delta u_i = \sum_n b_{ni}u_n \tag{11a}$$

$$F_i = \sum_n a_{ni}u_n \tag{11b}$$

However, according to Eq. (7),

$$a_{ii} = 0 \tag{12}$$

The remaining a_{ni} which satisfy $a_{ni} = (F_i,Mu_n)_\Omega$ are determined from Eq. (6b). Thus

$$a_{ni} = \lambda_i((\delta Mu_i,G)_\Omega,\ Mu_n)_\Omega - ((\delta ETu_i,TG)_\Omega,\ Mu_n)_\Omega \tag{13}$$

In deriving Eq. (13), the self-adjoint nature of δE and δM as well as Eq. (10) were used. Once again, substitution of Eq. (2) into the expression resulting from changing the orders of integration in Eq. (13) simplifies the coefficients to the following:

$$a_{ni} = \lambda_n^{-1}\{\lambda_i(u_n,\delta Mu_i)_\Omega - (Tu_n,\delta ETu_i)_\Omega\} \tag{14}$$

The coefficients b_{ni} are similarly determined by substituting Eq. (11a) into Eq. (6a). The result is

$$b_{ni} = \frac{\lambda_n}{\lambda_n - \lambda_i}a_{ni} \qquad i \neq n \tag{15}$$

The remaining coefficient b_{ii} is determined from the normality condition [Eq. (10)], expressed in the form

$$(Mu_i,\delta u_j)_\Omega + (Mu_j,\delta u_i)_\Omega = -(\delta Mu_i,u_j)_\Omega \tag{16}$$

Substitution of Eq. (11a) into Eq. (16) immediately implies

$$b_{ii} = -\tfrac{1}{2}(u_i,\delta Mu_i)_\Omega \tag{17}$$

As a result of Eqs. (13), (15), and (17), Eq. (11a) becomes

$$\delta u_i = \sum_{\substack{n=1\\n\neq i}}^{\infty}\frac{\lambda_i u_n}{\lambda_n - \lambda_i}(u_n,\delta Mu_i)_\Omega - \sum_{\substack{n=1\\n\neq i}}^{\infty}\frac{u_n}{\lambda_n - \lambda_i}(Tu_n,\delta ETu_i)_\Omega - \tfrac{1}{2}(u_i,\delta Mu_i)_\Omega u_i \tag{18}$$

It was pointed out earlier that the eigenvalue variation for buckling could be obtained by setting $M=1$, in which case, $\delta M = 0$; thus

$$\delta u_i = -\sum_{\substack{n=1\\n\neq i}}^{\infty}\frac{u_n}{\lambda_n - \lambda_i}(Tu_n,\delta ETu_i)_\Omega \tag{19}$$

is valid for buckling. Similarly, for composite fiber-reinforced laminates undergoing free vibration, δM will vanish if the design variables are the orientation of the constituent laminae. In this case, Eq. (19) will also apply.

Repeated Eigenvalues

Now assume that λ_i is a repeated eigenvalue, that is, there are two distinct orthonormal modes, u_i and u_{i+1}, corresponding to the same eigenvalue λ_i. If the eigenvalues are arranged in ascending order, then

$$\lambda_{i-1} < \lambda_i = \lambda_{i+1} < \lambda_{i+2} \tag{20}$$

The eigenmodes will still satisfy Eq. (10) for all values of the indices i and j. A solution to Eqs. (6a) and (6b) is sought, if it exists. In this case, Fredholm's Theorem states that u_i will exist if, and only if, both Eq. (7) and

$$(F_i, Mu_{i+1})_\Omega = 0 \tag{21}$$

are satisfied.

It has already been shown that Eq. (7) is satisfied, and therefore, so is Eq. (9). However, Eq. (6b), together with some routine calculations, show that Eq. (21) is equivalent to

$$\lambda_i(\delta Mu_i, u_{i+1})_\Omega = (\delta ETu_i, Tu_{i+1})_\Omega \tag{22}$$

It should be noted that Eq. (22) will be satisfied for only very select variations δS. Since Eq. (22) is not satisfied for all δS, it follows that Eqs. (6a) and (6b) do not, in general, admit a solution, and thus eigenfunctions corresponding to repeated eigenvalues are not differentiable with respect to the design parameters.

It is possible to restrict the variations δS so that Eq. (22) is satisfied. In this case, solutions for δu_i will exist, and these variations shall be called directional variations, analogous to the concept of directional derivatives.

Let the direction in the design space be such that

$$\lambda_i(S+\delta S) = \lambda_{i+1}(S+\delta S) \tag{23}$$

that is, the variation is required to preserve the bimodal character of the repeated eigenvalue. Let u be an arbitrary normalized linear combination of the eigenfunctions u_i and u_{i+1}. Thus u is also an eigenfunction corresponding to the common eigenvalue

$$\lambda_i = \lambda_{i+1} \equiv \lambda \tag{24a}$$

and

$$u = \cos\theta u_i + \sin\theta u_{i+1} \tag{24b}$$

Substitution of Eqs. (24) and (10) into Eq. (9) establishes the validity of Eq. (22). Consequently, the directional variations δu_i and δu_{i+1} exist, and are the solutions to

$$\delta u_i - \lambda(G, M\delta u_i)_\Omega = F_i \tag{25a}$$

$$\delta u_{i+1} - \lambda(G, M\delta u_{i+1})_\Omega = F_{i+1} \tag{25b}$$

where F_i and F_{i+1} are given by Eq. (6b).

The solutions to Eqs. (25) are obtained in the same way that Eqs. (6) were solved for distinct eigenvalues. In this case the directional variations δu_i and δu_{i+1} are unique only to within linear combinations of u_i and u_{i+1}. The four constants thus obtained are not completely arbitrary, since the solutions must meet the orthonormality conditions [Eq. (16)]. The results are expressed for the special case in which the lowest eigenvalue is repeated. Thus

$$\delta u_1 = -\tfrac{1}{2}(u_1, \delta Mu_1)u_1 + \epsilon u_2 + \sum_3^\infty \frac{\lambda u_n}{\lambda_n - \lambda}(u_1, \delta Mu_n) - \sum_3^\infty \frac{u_n}{\lambda_n - \lambda}(Tu_1, \delta ETu_n)$$

$$\delta u_2 = -\epsilon u_1 - (u_1, \delta Mu_2)u_1 - \tfrac{1}{2}(u_2, \delta Mu_2)u_2 + \sum_3^\infty \frac{\lambda u_n}{\lambda_n - \lambda}(u_2, \delta Mu_n) - \sum_3^\infty \frac{u_n}{\lambda_n - \lambda}(Tu_2, \delta ETu_n)$$

where ϵ is an arbitrary infinitesimal scalar.

Second Variation

Proceeding from Eq. (9), the second variation $\delta^2\lambda_i$ is readily found to be

$$\delta^2\lambda_i = 2(T\delta u_i, \delta ETu_i)_\Omega + (Tu_i, \delta^2 ETu_i)_\Omega - 2\lambda_i(\delta u_i, \delta Mu_i)_\Omega - \lambda_i(u_i, \delta^2 Mu_i)_\Omega - \delta\lambda_i(u_i, \delta Mu_i)_\Omega \tag{26}$$

After substituting δu_i from Eq. (18) into Eq. (26), and making use of Eq. (9), Eq. (26) simplifies to

$$\delta^2\lambda_i = (Tu_i, \delta^2 ETu_i)_\Omega - \lambda_i(u_i, \delta^2 Mu_i)_\Omega - 2\delta\lambda_i(u_i, \delta Mu_i)_\Omega - 2\sum_{n\neq i} \frac{[\lambda_i(u_n, \delta Mu_i)_\Omega - (Tu_n, \delta ETu_i)_\Omega]^2}{\lambda_n - \lambda_i} \tag{27}$$

For repeated eigenvalues, the computation is only slightly more complicated. Assuming the lowest eigenvalue is bimodal, then the second directional variation $\delta^2\lambda$ is still given by Eq. (26) but here, i is set equal to unity everywhere. Similarly, Eq. (22) also holds for $i=1$. In this case, by following a procedure similar to the development of Eq. (27), it can be shown that

$$\delta^2\lambda = (Tu_1, \delta ETu_1)_\Omega - \lambda(u_1, \delta^2 Mu_1)_\Omega - 2\delta\lambda(u_1, \delta Mu_1)_\Omega - 2\sum_3^\infty \frac{[\lambda(u_n, \delta Mu_1)_\Omega - (Tu_n, \delta ETu_1)_\Omega]^2}{\lambda_n - \lambda} \tag{28}$$

Application to Structural Optimization

Some of the more interesting problems in structural optimization concern maximizing the lowest eigenvalue in the design space. The desirability of maximizing either the critical buckling load or the fundamental frequency is self-evident. The papers[3-5,8,9] provide a representative sampling of the interest. In virtually all of the studies, the optimality condition is determined by appending the constraint functional

$$J(S) - J_0 \leq 0$$

to $\lambda_1(S)$ through the use of a Lagrange multiplier μ. Thus

$$\delta\lambda_1 - \mu\delta J = 0 \tag{29}$$

where $\delta\lambda_1$ is given by Eq. (9).

Once Eq. (29) is solved for the design, it is still necessary to ask two questions: 1) Does the design found indeed make the lowest eigenvalue stationary or is some other eigenvalue stationary instead? 2) If the lowest eigenvalue is stationary, is it a local maximum in the design space? For sandwich-type structures, where both E and M are linear in the design variable, Prager and Taylor[3] answered both questions in the affirmative by making use of the extremal character of Rayleigh's quotient. For other types of structures, the answer to these questions has been somewhat more elusive. Indeed, Olhoff and Rasmussen[9] recently inspired much research into repeated eigenvalues by showing that the prior work of Tadjbakhsh and Keller[8] merely determined a stationary design for λ_2 and that the lowest eigenvalue was repeated at the optimum design.

Whether or not a structural optimality condition is also a sufficient condition may be established from Eq. (27) if the lowest eigenvalue is unimodal, or Eq. (28) if it is bimodal. Suppose that a design S_0 is selected so that

$$\delta\lambda_1 |_{S=S_0} = 0 \tag{30}$$

for all infinitesimal variations δS consistent with the constraints. The term $\delta\lambda_1$ refers to the usual first variation if the eigenvalue is unimodal and to the directional first variation if the eigenvalue is bimodal. In either case, it follows from Eq. (30) that, to lowest order,

$$\Delta\lambda_1 = \delta^2\lambda_1$$

where $\delta^2\lambda_1$ refers either to the second variation [Eq. (27)] or the directional second variation [Eq. (28)]. A sufficiency condition for optimality is, therefore,

$$\delta^2\lambda_1 < 0 \tag{31}$$

There are several general cases for which it can be shown that inequality [Eq. (31)] is satisfied whenever Eq. (30) is satisfied. First, suppose that M and E are linear in the design variables. Then $\delta^2 M$ and $\delta^2 E$ vanish identically, and it is clear from Eqs. (27) and (28) that inequality [Eq. (31)] is satisfied for either a distinct or repeated lowest eigenvalue. Consequently, Eq. (30) is both a necessary and sufficient condition for optimality. Next, suppose that M is independent of S. In this case, Eq. (27) becomes

$$\delta^2\lambda_1 = (Tu_1, \delta^2 ETu_1)_\Omega - 2\sum_2^\infty \frac{(Tu_n, \delta ETu_1)_\Omega^2}{\lambda_n - \lambda_i} \tag{32}$$

A similar expression is obtained if Eq. (28) is used. In many cases, it is possible to select the design variable so that $\delta^2 E$ vanishes. For example, suppose that the structure under consideration is a rod with variable circular cross-section. It is desired to determine the radius r in terms of the position x for, say, a fixed volume constraint. It is clear that, without loss in generality, S may be equated with r, r^2 (area), or r^4 (second moment of area). Since the stiffness E is proportional r^4, the latter choice renders the problem linear. Thus, it is evident that $\delta^2\lambda_1 < 0$, and, once again, Eq. (30) is both necessary and sufficient for optimality.

In the general case, it is not clear whether the necessary condition [Eq. (30)] for optimality is also a sufficient condition. In the absence of a direct proof to the contrary, it appears necessary to determine the sign of $\delta^2\lambda$ on a case by case basis.

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